MAT224: Differential Equations
Existence and Uniqueness of Solutions

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Consider the Initial Value Problem:

$$y'(x) = f(x, y) \qquad y(x_0) = y_0.$$

It would be good to know if a solution to this IVP exists before going to look for it. So we devote some time now to that issue. The main tool that gives us sufficient conditions under which we can be guaranteed a (unique) solution is *Picard's Theorem*:

Theorem 1 (Picard)**.** *Let $f(x, y)$ and $\partial f/\partial y$ be continuous functions of x and y on a closed rectangle R with sides parallel to the x and y axes. If (x_0, y_0) is on the interior of R, then there exists $h > 0$ with the property that the initial value problem*

$$y'(x) = f(x, y) \qquad y(x_0) = y_0$$

has a unique solution on the interval $|x - x_0| \leq h$.

Sketch of Proof. First, we convert to an *integral equation*:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y)\, \mathrm{d}t.$$

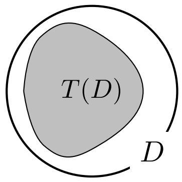
We wish to find a continuous solution $y(x)$ to the above *integral equation*. Let $R = [a, b] \times [c, d]$, and equip $C[a, b]$, the Vector Space of continuous functions on $[a, b]$, with the norm

$$\|g\| = \max_{x \in [a,b]} |g(x)|.$$

Consider the operator $T : C[a, b] \to C[a, b]$ defined by

$$T(g) = y_0 + \int_{x_0}^{x} f(t, g)\, \mathrm{d}t.$$

The main idea is that the hypotheses on the continuity of f and its partial derivative force the operator T to be a *contraction* as viewed on $[x_0 - h, x_0 + h]$ for h sufficiently small:



This means that there exists a constant $K < 1$ such that $\|T(g_1) - T(g_2)\| \leq K\|g_1 - g_2\|$:

A contracting map must have a *unique* fixed point. In this case, this means there exists a function $y(x)$ such that $T(y) = y$. This is the solution to our differential equation. $\square$

Example: For what points (x_0, y_0) does Picard's Theorem guarantee that the initial value problem

$$y' = y \cdot |y|, \qquad y(x_0) = y_0,$$

has a unique solution on some interval $|x - x_0| \leq h$?

Solution

Note that $f(x, y) = y \cdot |y|$ is continuous everywhere. Also note that

$$\frac{\partial f}{\partial y} = \begin{cases} 2y & y > 0 \\ 0 & y = 0 \\ -2y & y < 0. \end{cases}$$

So $\partial f/\partial y$ is continuous everywhere as well. So for *every* point (x_0, y_0), the IVP $y' = y \cdot |y|,\ y(x_0) = y_0$ has a unique solution on some interval containing x_0.

HW: Due Wednesday, 15-Jun

1. For what points (x_0, y_0) does the ODE
$$y' = \frac{y^2}{1 + x^2}$$
have a solution passing through (x_0, y_0)?

2. For what points (x_0, y_0) does the ODE
$$y' = \frac{y - x^2}{y^2 - x}.$$
have a solution passing through (x_0, y_0)?

3. Does an Etch-A-Sketch exist? Etch-A-Sketch is (allegedly) a drawing mechanism where a curve is drawn by toying with two knobs that continuously vary curvature. So, in order for an Etch-A-Sketch to exist, the graph of any function $y = y(x)$ must be drawable by such means. Is this true? Is a curve uniquely determined by *how* it curves?

   The curvature of a curve $y = y(x)$ at $(x, y(x))$ is given by
$$\kappa(x) = \frac{y''(x)}{(1 + (y'(x))^2)^{3/2}}.$$
   The goal in this problem to try to recover $y(x)$ if we are given $\kappa(x)$.

   (a) Let $Y = y'$. Under what conditions (on $\kappa(x)$ or (x_0, y_1) if any) can we guarantee that there exists a unique solution to
$$Y' = \kappa(x) \cdot (1 + Y^2)^{3/2}$$
   passing through the point (x_0, y_1)?

   (b) Find an explicit solution to the ODE
$$Y' = \kappa(x) \cdot (1 + Y^2)^{3/2}$$
   subject to the initial condition $Y(x_0) = 0$. Your answer will be in terms of the function
$$\mathrm{K}(x) = \int_{x_0}^{x} \kappa(t)\, \mathrm{d}t.$$

   (c) Find a function $y(x)$ satisfying $y'(0) = 0$ and $y(0) = 1$ whose curvature is $\kappa(x) = \cos(x)$ on $[0, \pi/2)$.